B.SC IST SEMISTER (PHYSICS)
2016

Momentum and Energy:

By our prehand knowledge we are aware that in dynamics we are concerned with Mechanical Energy (ME) which includes the Kinetic Energy (KE) and the Potential Energy (PE).

It is further known to us that the KE is possessed by virtue of motion and is given by:

$$KE = \frac{p^2}{2m}$$

where *p* is the momentum of the body of mass "m".

P Characterizes the quantity of motion possessed by a body of mass "m", moving with velocity *V* , so

$p = mV$

If an object is not in motion, then the energy possessed by it will obviously be by virtue of its position and configuration. This is called PE, e.g., for a body of mass "m" standing at height "h" against Earth's gravity,

$PE = mgh$

Conservation of Linear Momentum

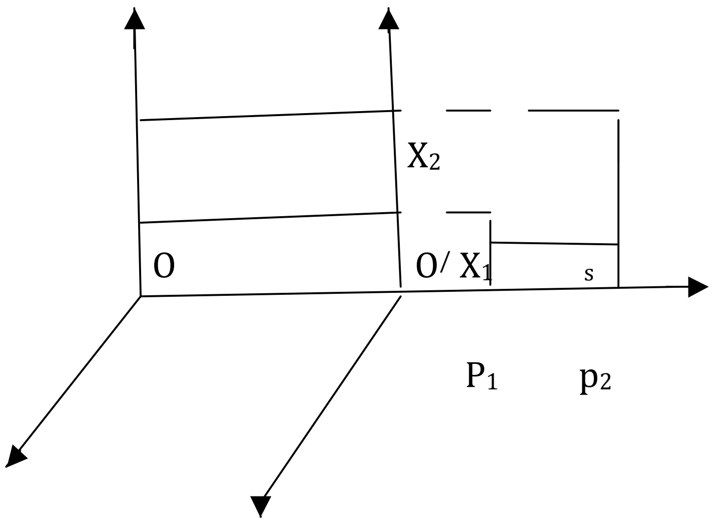
Mathematically, when a force *F* is applied to a moving body its momentum *p* changes at the rate $\frac{dp}{dt}$, so :

$$\vec{F} = \frac{d\vec{p}}{dt}$$ (Newton's2ndlawofmotion):

Conservation of linear momentum follows as an immediate consequence of Newton's 2nd law of motion,

As $F = 0$, p= Constant

The same result can be re-affirmed by an alternative approach, for which we shall first introduce the concept of "Linear Uniformity of Space" (LUS), according to which the result of an experiment is not altered due to linear displacement of the co-ordinate system.



In other words space is homogeneous and possesses translational invariance. This also demands that the force between two particles should be independent of the displacement between the two frames of reference.

Let the two particles P_1 and P_2 at positions x_1 and x_2 along x-axis, be at positions x_1'andx_2' after displacement of co-ordinate system.

It is easy to see that

$$x_2 - x_1 = x_2' - x_1' = S \tag{1}$$

Now expressing force (F) as the gradient of potential,

$$F = -\frac{\partial U}{\partial x}$$

$$F_{21} = -\frac{\partial U}{\partial x_1} = -\frac{\partial U}{\partial S}\frac{\partial S}{\partial x_1} = +\frac{\partial U}{\partial S}$$

$$F_{12} = -\frac{\partial U}{\partial x_2} = -\frac{\partial U}{\partial S}\frac{\partial S}{\partial x_2} = -\frac{\partial U}{\partial S}$$

As $\{\frac{\partial S}{\partial x_1} = -1, \quad \frac{\partial S}{\partial x_2} = 1\}$ (From Eq. 1)

Therefore, $F_{12} = -F_{21}$ (Newton's 3rd Law of Motion)

Or

$F12 + F21 = 0$

Total Force, $F = 0$

$\Rightarrow p = Constant$

Thus the uniformity of space proves the law of conservation of linear momentum with the help of Newton's 3rd law of motion.

Work and Energy

If the forces are derivable from scalar PE function in the manner, F = −∇ U Then under the action of such a force, the total energy of the particle (KE + PE) is conserved. Suppose the particle moves from position 1 to position 2, then:

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} \tag{2}$$

$$\begin{aligned} W_{12} &= \int_1^2 \frac{\partial U}{\partial r} d\vec{r} \\ &= -\int_1^2 \partial U \\ &= U_1 - U_2 \end{aligned} \tag{3}$$

Equivalently Eq. 1 can also be written as

$$W_{12} = \int_1^2 \frac{d\vec{p}}{dt} dr, \tag{4}$$

$$\dot{r} = \frac{dr}{dt}, \quad dr = \dot{r} dt$$

Using above equations and also

$$p = m\dot{r}$$

$$\begin{aligned} W_{12} &= \int_1^2 \frac{d}{dt}(m\dot{r}).\dot{r} dt \\ &= \int_1^2 \frac{d}{dt}(\frac{1}{2}m\dot{r}^2) dt \\ &= K_2 - K_1 \end{aligned} \tag{5}$$

From Equatios. 3 and 5

$U_1 - U_2 = K_2 - K1$

U_s and K_s denote potential energies kinetic energies respectively

$U_1 + K_1 = U_2 + K1$

$E1 = E2$

which shows that the Total Energy of the particle is conserved.

Conservation of Energy for a System of Particles

When a mechanical system consists of two or more particles, we must distinguish between the external forces exerted upon the particles by the sources not belonging to the system and the internal forces arising on account of mutual interaction

For mutual interaction between any two particles i, j the scalar potential function for the whole system is given as:

$$V^{int} = \frac{1}{2}\sum_{i,j} V_i^j \qquad (6)$$

a factor $\frac{1}{2}$ comes because of the fact, that while summing the internal potential energies a pair of particles i, j appears twice.

For external forces, the scalar potential function of the system is given as:

$$V\,ext = \sum vi$$

The two can be summed up as

$$V = V\,ext + V\,int$$

Such that

$$F_i^{ext} = -\nabla_i V^{ext}$$

$$F_i^{int} = -\nabla_i V^{int}$$

The Kinetic Energy of the system of the particles is given as:

$$KE = \sum_i \frac{1}{2} m_i \dot{r}_i^2$$

Now suppose that the ith particle is displaced through dr_i, then the amount of the work done by the total force F_i acting on the ith particle is $F_i dr_i$ which can be expressed as:

$$m_i \ddot{r}_i . dr_i = F_i . dr_i \qquad (7)$$

using

$$\frac{dr_i}{dt} = \dot{r}$$

and

$$F_i = F_i^{ext} + F_i^{J\ int}$$

Eq. 7 can be written as

$m_i \ddot{r}_i . \dot{r}_i dt = F_i^{ext} dr_i + F_i^{J\ int} dr_i$

Or

$\frac{d}{dt}(\frac{1}{2} m_i \dot{r_i}^2) dt = F_i^{ext} dr_i + F_i^{J\ int} dr_i$

For the whole system of particles

$\frac{d}{dt}(\sum_i \frac{1}{2} m_i \dot{r_i}^2) dt = \sum_i F_i^{ext} dr_i + \sum_{ij} F_j^{i\ int} dr_i$

Integrating both sides we get

$\sum_i \frac{1}{2} m_i \dot{r_i}^2 = -\sum_i V^{ext} - \frac{1}{2} \sum_{ij} V_i^{J\ int} + c$

where "c" is constant of integration

KE	= −*P.E* + *Constant*
KE + *P.E*	= *Constant*
Total Energy	= *Constant*

Motion of Rockets

Rocket is a device which is used to place a satellite into an orbit. Rocket propulsion is based on the principal of conservation of momentum. A rocket carries both fuel and oxidizes which burn in the combustion chamber within the Rocket. When the rocket is fired the exhaust gases rushes downwards at a high speed and push the rocket upward.

The differences of the two forces gives the net force acting on the system

$$\vec{F} = M \frac{d\vec{V}}{dt} - \vec{V}_e \frac{dM}{dt} \quad (8)$$

where "M" is the mass of the rocket along with its fuel and oxidiser. "dM" is the mass of the exhaust gases ejecting with a velocity $V^{\sim}_e$.

Near the earth, the upward thrust on the rocket is opposed by the force of gravitation

F= *Mg*

Therefore Eq. 8 becomes:

$$\begin{aligned} M\vec{g} &= M\frac{d\vec{V}}{dt} - \vec{V}_e\frac{dM}{dt} \\ M\frac{d\vec{V}}{dt} &= V_e\frac{dM}{dt} + M\vec{g} \end{aligned} \quad (9)$$

Outside the influence of gravity Eq. 10 reduces to:

$$\begin{aligned} M\frac{d\vec{V}}{dt} &= \vec{V}_e\frac{dM}{dt} \\ d\vec{V} &= V_e\frac{dM}{M} \end{aligned} \quad (10)$$

Integrating Eq. 11 b/w necessary limits:

$$\begin{aligned} \int_{V_0}^{V} &= V_e\int_{M_0}^{M}\frac{dM}{M} \\ \vec{V} - \vec{V}_0 &= \vec{V}_e log\frac{M}{M_0} \\ \vec{V} &= \vec{V}_0 + \vec{V}_e log\frac{M}{M_0} \end{aligned}$$

Where M_0 and V_0 are initial mass and initial velocity of the rocket. If the rocket starts from the rest, $V_0 = 0$.

$$\vec{V} = V_e\ log\frac{M}{M_0}$$

Or

$$\vec{V}_f = V_e\ log\frac{M}{M_0}$$

This is called Maximum or burnt out velocity of the rocket when the entire fuel has been exhausted.

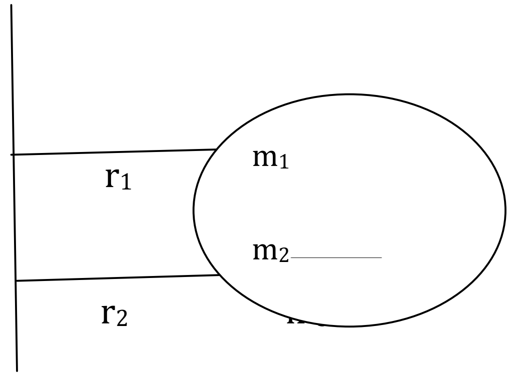
Rotational Motion

When the body moves such that its distance from a fixed point (or axis) remains constant, it is said to possess rotational motion. Take for instance the different constituent particles of the rigid rotating body consequently, the definition of every parameter of motion gets modified in terms of this vector distance ~r from the fixed point in the inertial frame.

Moreover, during the rotatory motion, the body keeps on sweeping different angles, so changes the nomenclature as follows

Angular velocity denoted by ω or $\dot{\theta}$

Angular momentum denoted by Lor J

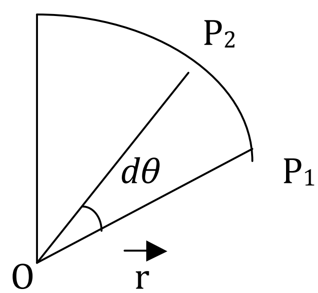


Turning force called Torque and is denoted as τ (tau) All these terms are explained ahead.

Angular Velocity:

It is given by the rate of change of angular displacement

$\omega = \dot{\theta} = \frac{d\theta}{dt}$



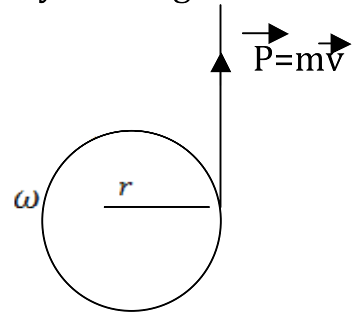
where $d\theta$ is the small angle subtended by a circularly moving particle at the fixed point while moving from the position P_1 to the position P_2. If the length of arc $P_1P_2 = l$, then

$l = rd\theta$

By differentiating w.r.t time

$V = r\omega$ (11)

which gives the relationship b/w linear velocity and angular velocity.



Similarly, angular momentum is given by the moment of linear momentum about the fixed point

$L = r \times p = r \times mV = mr^2\omega$ (By using Eq. 11).

Torque

Torque is turning effect of force. If a force F acts on the particle at point p whose position w.r.t the origin "O" of the inertial frame of reference is given by r, then torque:

$$\tau = r \times F \quad (12)$$

Torque is also expressed as the time rate of change of angular momentum J

$$\vec{J} = \vec{r} \times \vec{P} \quad (13)$$

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\frac{d\vec{J}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r}\frac{d\vec{p}}{dt}$$

Now,

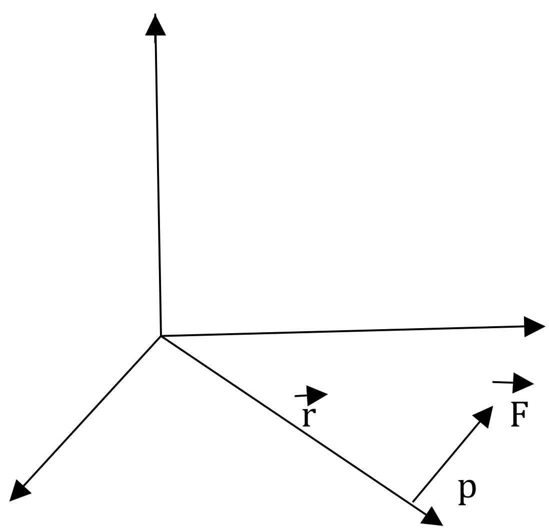
$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{V} \times m\vec{V} = m(\vec{V} \times \vec{V}) = 0$$

Also

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Therefore,

$$\frac{d\vec{J}}{dt} = \vec{r} \times \vec{F} \quad (14)$$



From Eq. 12 and Eq. 14, we get

$$\tau = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$$

Conservation of Angular Momentum

There are two ways of starting it:

We know that

$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

when the total external torque (τ) acting on the system is zero, the total angular momentum *J* remains constant.

Again as

$\tau = r \times F$

If the particle is subjected to central force depending upon the distance from a fixed point, then

$F = f(r)\ \hat{r}$

$$\vec{\tau} = \vec{r} \times f(r)\ \hat{r} = f(r)(\vec{r} \times \hat{r}) = f(r)(\vec{r} \times \frac{\vec{r}}{|\vec{r}|}) = 0$$

By definition, a unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$. Hence the angular momentum of a particle under the influence of a central force remains constant.